Math 142 Lecture 11 Notes

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February 20, 2018

1 Review: Identification Spaces and Embeddings

1.1 Identification spaces and continuity

1.1.1 Identification spaces and attaching maps

Let's review the concept of an identification space.

Let X be a space with a partition $\mathcal{P}$ of X. We have a function $p : X \to \mathcal{P}$ mapping x to the element of $\mathcal{P}$ containing x. Define a space Y that has:

- Points are elements of $\mathcal{P}$.
- Open sets are $U \subseteq \mathcal{P}$ such that $p^{-1}(U)$ is open.

Example 1.1. Let $X = \{1, 2, 3, 4\}$ with the open sets $\{\varnothing, \{1\}, \{3\}, \{1, 3\}, X\}$. Let

$$\mathcal{P} = \{\{1, 4\}, \{2, 3\}\}.$$

Then $p(1) = \{1, 4\}$, $p(2) = \{2, 3\}$, $p(3) = \{2, 3\}$, and $p(4) = \{1, 4\}$.

What sets are open in Y? We have $p^{-1}(\varnothing) = \varnothing \subseteq X$, so $\varnothing$ is open in Y. Similarly, $p^{-1}(\{\{1, 4\}, \{2, 3\}\}) = X$ is open, so the whole space Y is open. However, $p^{-1}(\{1, 4\}) = \{1, 4\} \subseteq X$ is not open, so $\{1, 4\}$ is not open in Y. Also, $p^{-1}(\{2, 3\}) = \{2, 3\} \subseteq X$ is not open, so $\{2, 3\}$ is not open in Y.

So we can call this space $Y = \{a, b\}$ with open sets $\{\varnothing, Y\}$, where $a = \{1, 4\}$ and $b = \{2, 3\}$.

The function $p : X \to \mathcal{P}$ corresponds to a map $p : X \to Y$. Is p continuous? If $U \subseteq Y$ is open, then $p^{-1}(U) \subseteq X$ is open; so yes, p is continuous. In general, this is not the only topology for which p is continuous, but it is the largest such topology.

Theorem 1.1. *If X is a space, Y is an identification space (created from X), and Z is another space with maps*

$$X \xrightarrow{p} Y \xrightarrow{f} Z,$$

then f is continuous iff $f \circ p$ is continuous.

Proof. This follows straight from the definitions of continuity and the topology on Y.

$$\begin{aligned} f \text{ is continuous} &\iff \forall U \subseteq Z \text{ open}, f^{-1}(U) \subseteq Y \text{ is open} \\ &\iff \forall U \subseteq Z \text{ open}, p^{-1}(f^{-1}(U)) \subseteq X \text{ is open} \\ &\iff \forall U \subseteq Z \text{ open}, (f \circ p)^{-1}(U) \subseteq X \text{ is open} \\ &\iff f \circ p \text{ is continuous.} \end{aligned}$$

□

1.1.2 The largest topology with respect to continuity

Here is question 1c from the 2016 midterm.

Let $X = \{1, 2, 3, 4, 5\}$ with the topology with the base $\{\{1\}, \{1, 2\}, \{3\}, \{4, 5\}\}$. Let $f : X \to \{a, b, c\}$ be

$$f(1) = f(3) = a, \qquad f(2) = f(4) = b, \qquad f(5) = c.$$

What is the largest topology on Y such that f is continuous?

We want $U \subseteq Y$ open iff $f^{-1}(U) \subseteq X$ open. Let's check a few sets:

- $f{-}1(\{a\}) = \{1, 3\}$ is open, so $\{a\}$ is open.
- $f^{-1}(\{c\}) = \{5\}$ is not open, so $\{c\}$ is not open.
- $f^{-1}(\{b\}) = \{2, 4\}$ is not open, so $\{b\}$ is not open.
- $f^{-1}(\{a, c\}) = \{1, 3, 5\}$ is not open, so $\{a, c\}$ is not open.
- $f^{-1}(\{b, c\}) = \{2, 4, 5\}$ is not open, so $\{b, c\}$ is not open.
- $f^{-1}(\{a, b\}) = \{1, 2, 3, 4\}$ is not open, so $\{a, b\}$ is not open.

So the largest topology on Y making f continuous is $\{\varnothing, \{a\}, Y\}$.

1.2 Embeddings

Definition 1.1. An *embedding* $f : X \to Y$ is a function such that if we consider this as a map $f : X \to f(X)$, then f is a homeomorphism. Here, $f(X)$ has the subspace topology.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}^2$ send $x \mapsto (x, 0)$. Then f is an embedding of the real line into the plane.

Example 1.3. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $f : \mathbb{R} \to \mathbb{R}^2$ sending $x \mapsto (x, g(x))$ is an embedding sending x to the *graph* of x.

Example 1.4. The following is not an embedding. Let $f : [0,1) \to \mathbb{C}^2$ send $x \mapsto e^{2\pi i x}$. Here, f is a continuous bijection onto its image, the unit circle in $\mathbb{C}$. However, this is not a homeomorphism because $[0,1/2)$ is open in the subspace topology on $[0,1)$, but $f([0,1/2])$ is not open in $S^1 \subseteq \mathbb{C}$.

How do we make an embedding in this case? First, let $f : [0,1] \to \mathbb{C}$ be $f(x) = e^{2\pi i x}$. However, this is not injective, so we use an identification space. Define the partition on $[0,1]$: $\mathcal{P} = \{\{x\} : x \neq 0,1\} \cup \{\{0,1\}\}$ The identification space Y is homeomorphic to S^1. We showed this in class ($B^1/S^0 \cong S^1$).

So we get an induced map $\tilde{f} : Y \to \mathbb{C}$, where $\{x\} \mapsto f(x)$, $\{0,1\} \mapsto f(0) = f(1)$, and $f(x) = \tilde{f}(p(x))$ for all $x \in [0,1]$

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "p"] & \mathbb{C} \\ S^1 \cong Y \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Here, $\tilde{f}$ is continuous iff f is continuous. We have $\tilde{f} : Y \to \mathbb{C} \cong \mathbb{R}^2$, where the domain is compact (as the continuous image of a compact space) and the codomain is Hausdorff (as a metric space), so f is a homeomorphism.